Assignment 6.

Complex integral.

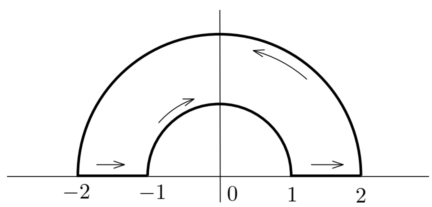
This assignment is due Wednesday, March 4. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

Note that these problems do not require Cauchy Theorem. (Except, in Problem 3 you can, but do not have to, use it.)

(1) Evaluate the integrals $J_1 = \int_L x\,dz$, $J_2 = \int_L y\,dz$, $J_3 = \int_L |z|\,dz$ along the following curves (*Hint:* use parametrization):
  (a) The line segment joining points $z = 0$ and $z = 2 + i$,
  (b) The semicircle $|z| = 1$, $\mathrm{Im} z \geq 0$, with initial point $z = 1$,
  (c) The circle $|z - a| = R$. (The $\int_L |z|\,dz$ is optional for this one).

(2) Evaluate the integral
$$\int_L \frac{z}{\bar{z}} dz,$$
where L is a closed contour that bounds "upper semi-ring" $1 \leq |z| \leq 2$, $\mathrm{Im}\, z \geq 0$, traversed counterclockwise (see figure). (*Hint:* The answer is $4/3$).

Why this integral being nonzero does not contradict Cauchy Theorem?



(3) Evaluate the integral
$$\int_{|z-a|=R} (z-a)^n dz$$
($R > 0$) for all values of the integer n.

(4) Prove that
$$\lim_{r \to 0} \int_{|z-a|=r} \frac{f(z)}{z-a} dz = 2\pi i f(a),$$
if f is continuous in a neighborhood of the point $z = a$.

(*Hint:* Parametrize the path of integration as $z = a + re^{it}$. If f is continuous at $z = a$, you can write $f(z) = f(a) + (f(z) - f(a))$, and $f(z) - f(a) \to 0$ as $z \to a$, in particular, as $r \to 0$.)

(5) Prove that if $f(z)$ is continuous in the closed domain $|z| \geq R_0$, $0 \leq \arg z \leq \alpha$ ($0 \leq \alpha \leq 2\pi$), and if the limit
$$\lim_{z \to \infty} z f(z) = A$$
exists, then
$$\lim_{R \to \infty} \int_{\Gamma_R} f(z) dz = iA\alpha,$$
where Γ_R is the arc of the circle $|z| = R$ lying in the given domain.

(*Hint:* Parametrize the arc by Re^{it}. Similarly to the previous problem, write $f(z) = \frac{A + (zf(z) - A)}{z}$ in the integral.)